(RE)USING DUAL INFORMATION IN MILP

INFORMS 2014, San Francisco

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AGENDA

1. LP Duality
2. Reusing Dual Information
3. Computational Results

- Consider the following LP problem

$$
\begin{aligned}
z = \min\ & cx \\
\text{s.t. } & L \le Ax \le U \\
& \ell \le x \le u
\end{aligned}
$$

- Assume z is finite. Any *dual solution* y satisfies (all terms are finite):

$$
\begin{aligned}
z \ge & \sum_{i: y_i > 0} L_i y_i + \sum_{i: y_i < 0} U_i y_i \quad + \sum_{i: c_i - A_i^T y > 0} \ell_i (c_i - A_i^T y) + \sum_{i: c_i - A_i^T y < 0} u_i (c_i - A_i^T y) \\
= & z_y(L, U, \ell, u)
\end{aligned}
$$

- There exists an optimal dual solution y^* that satisfies $z = z_y^*(L, U, \ell, u)$

FARKAS LEMMA

- The following system is infeasible:

$$\begin{aligned} L \leq Ax \leq U \\ \ell \leq x \leq u \end{aligned}$$

- There is a dual ray r to prove it:

$$z_r(L, U, \ell, u) =$$

$$\sum_{i:r_i>0} L_i r_i + \sum_{i:r_i<0} U_i r_i + \sum_{i:-A_i^T r>0} \ell_i(-A_i^T r) + \sum_{i:-A_i^T r<0} u_i(-A_i^T r) > 0$$

DUAL INFORMATION IN BRANCH AND BOUND

Node is primal feasible/dual simplex hits objective limit:

- A dual solution y is revealed
- Globally valid and can be used elsewhere in tree
- Store dual solutions and use them later as needed
  - Prune nodes by bound
  - Fix/Tighten bounds
  - Variable selection

DUAL INFORMATION IN BRANCH AND BOUND

Node is infeasible:

- A dual ray r is revealed
- Store dual rays and use them later as needed
  - Prune nodes by infeasibility
  - Fix/Tighten bounds
  - Branch towards infeasibility
- Great for feasibility instances
- No upper bound/incumbent needed for dual ray tightening
- vs Conflict analysis:
  - Cheap to implement/use
  - Source of information is kept, no need to extract conflict clauses
  - Dynamic and adaptive

REUSING DUAL INFORMATION

NODE PRUNING

Using dual solution y:

- y is globally valid and gives a lower bound (weak duality) at any other node
- Compute dual objective value $z_y(L, U, \ell, u) =$

$$\sum_{i:y_i>0} L_i y_i + \sum_{i:y_i<0} U_i y_i + \sum_{i:c_i - A_i^T y>0} \ell_i (c_i - A_i^T y) + \sum_{i:c_i - A_i^T y<0} u_i (c_i - A_i^T y)$$

- Let $\bar{z}$ be upper bound/incumbent objective value
- Prune node if $z_y(L, U, \ell, u) \geq \bar{z}$

REUSING DUAL INFORMATION

NODE PRUNING

Using dual ray r:

- r can be used to detect infeasible nodes
- Compute ray objective value $z_r(L, U, \ell, u) =$

$$\sum_{i:r_i>0} L_i r_i + \sum_{i:r_i<0} U_i r_i + \sum_{i:-A_i^T r>0} \ell_i(-A_i^T r) + \sum_{i:-A_i^T r<0} u_i(-A_i^T r)$$

- Prune node if $z_r(L, U, \ell, u) > 0$
- What if $z_y < \bar{z}$ and $z_r \leq 0$?

REUSING DUAL INFORMATION

COLUMN BOUND TIGHTENING

- Using dual solution y: Reduced cost fixing
  $z_y(L, U, \ell, u) =$

$$\sum_{i: y_i > 0} L_i y_i + \sum_{i: y_i < 0} U_i y_i + \sum_{i: c_i - A_i^T y > 0} \ell_i (c_i - A_i^T y) + \sum_{i: c_i - A_i^T y < 0} u_i (c_i - A_i^T y)$$

- If $c_i - A_i^T y > 0$, then increasing ℓ_i also increases z_y
- If there is $\ell_i < \ell_i' < u_i$ such that

$$z_y(L, U, \ell, u) + (\ell_i' - \ell_i)(c_i - A_i^T y) \geq \bar{z},$$

  then we can tighten u_i to ℓ_i' where

$$\ell_i' = \ell_i + \frac{\bar{z} - z(L, U, \ell, u)}{c_i - A_i^T y}$$

  knowing that there are no better feasible solutions where $x_i > \ell_i'$

REUSING DUAL INFORMATION

COLUMN BOUND TIGHTENING

- Using dual ray r: Dual ray fixing
  $z_r(L, U, \ell, u) =$

$$\sum_{i:r_i>0} L_i r_i + \sum_{i:r_i<0} U_i r_i + \sum_{i:-A_i^T r>0} \ell_i(-A_i^T r) + \sum_{i:-A_i^T r<0} u_i(-A_i^T r)$$

- If $-A_i^T r > 0$, then increasing ℓ_i also increases z_r
- If there is $\ell_i < \ell_i' < u_i$ such that

$$z_r(L, U, \ell, u) + (\ell_i' - \ell_i)(-A_i^T r) > 0,$$

  then we can tighten u_i to ℓ_i' where

$$\ell_i' = \ell_i + \frac{-z_r(L, U, \ell, u)}{-A_i^T r}$$

  knowing that there are no feasible solutions where $x_i > \ell_i'$
- Note that we don't need an upper bound

REUSING DUAL INFORMATION

ROW BOUND TIGHTENING

- Same as reduced cost fixing/dual ray fixing applied to row bounds
- For $y_i > 0$, U_i to L_i' where

$$L_i' = L_i + \frac{\bar{z} - z_y(L, U, \ell, u)}{y_i}$$

- For $r_i > 0$, U_i to L_i' where

$$L_i' = L_i + \frac{-z_r(L, U, \ell, u)}{r_i}$$

REUSING DUAL INFORMATION

VARIABLE SELECTION

- Estimate child node objective values using dual solution y
- Assume we branch on x_i that takes a fractional value in current node LP: $x_i = f$
- Let Z_L ($[\ell_i, \lfloor f \rfloor]$) and Z_R ($[\lceil f \rceil, u_i]$) be the LP objective values of left and right child nodes
- If $c_i - A_i^T y < 0$:
  $$Z_L \geq \sigma_L^i = z_y(L, U, \ell, u) + (\lfloor f \rfloor - u_i)(c_i - A_i^T y)$$
- If $c_i - A_i^T y > 0$:
  $$Z_R \geq \sigma_R^i = z_y(L, U, \ell, u) + (\lceil f \rceil - \ell_i)(c_i - A_i^T y)$$
- Choose the variable that maximizes σ_L^i or σ_R^i

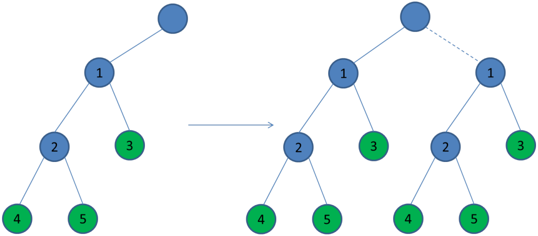
REUSING DUAL INFORMATION

VARIABLE SELECTION

- Estimate how close the child nodes are to be infeasible using dual ray r
- If $-A_i^T r < 0$:
  $\delta_L^i = z_r(L, U, \ell, u) + (\lfloor f \rfloor - u_i)(-A_i^T r)$
- If $-A_i^T r > 0$:
  $\delta_R^i = z_r(L, U, \ell, u) + (\lceil f \rceil - \ell_i)(-A_i^T r)$
- Choose the variable that maximizes δ_L^i or δ_R^i

REUSING DUAL INFORMATION | SUBTREES - IP BOUND

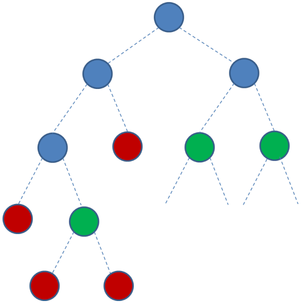
- Branch on x_i. Left child $[\ell_i, \lfloor f \rfloor]$, right child $[\lceil f \rceil, u_i]$



- Dual solutions: $y^1, \ldots, y^5$
- LP bound: $Z_R \geq \max_j\{z_{y^j}(L, U, \ell, u) + (\lceil f \rceil - \ell_i)(c_i - A_i^T y^j)\}$
- IP bound:
  - z_j: the dual objective value of node j with variable bounds of x_i replaced with $[\lceil f \rceil, u_i]$
  - $P_R \geq \max\{z_1, \min\{z_3, \max\{z_2, \min\{z_4, z_5\}\}\}\}$

DUAL BRANCHING

- Simulate branching with a pool of dual solutions and dual rays



- Select branching variable using dual information
- Evaluate child nodes
  - Prune if proven infeasible by a dual ray
  - Prune by bound by a dual solution
- Otherwise, select a child node and branch

COMPUTATIONAL RESULTS

SAS/OR

- 500 test instances (internal + public)
- 1h time limit
- Variable selection: 15% speedup + 15 instances
- Node pruning + bound tightening: 7% speedup + 8 instances

http://support.sas.com/or

(Re)using dual information in MILP